Mathematics Methods

Unit 3 & 4

Random variables

1. Random variables

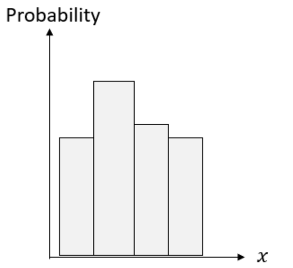
Definition: A variable whose possible values are outcomes of a random phenomenon

(a) Discrete random variable

Definition: a variable whose values (countable values) are obtained by counting

Example:

- Number of cats in SPCA
- Number of marbles in a jar
- Number of staffs in an office



(i) Probability distribution

Definition: is a list of all of the possible outcomes of a random variable along with their corresponding probability.

Probability distribution properties (characteristics):

- Probabilities for each value of X lies in the interval of $0 \leq P(X = x) \leq 1$
- Sum (total) of the probabilities is 1

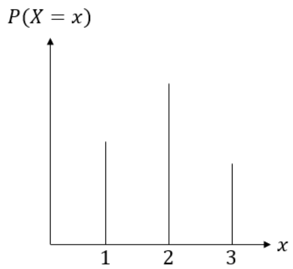
(x can be negative but $P(X = x)$ cannot be negative)

Probability distribution (discrete random variable) can be given by:

- Table form

x							
$P(X = x)$							

- Graphical form



- Function form

$$P(x) = P(X = x)$$

Probability of a discrete random variable by choosing with replacement

$P(X = x) = {}^{n}C_{r}(p)^{r}(1-p)^{n-r}$

Example:
A bag of chips contains 4 red and 2 blue chips. Three chips are drawn randomly without replacement. Draw a probability distribution for X: number of blue chips drawn.

Probability of a discrete random variable by choosing without replacement

$$P(X = x) = \frac{{}^{n_1}C_{r}\,{}^{n_2}C_{n-r}}{{}^{n_1+n_2}C_{n}}$$

Example:
A bag of marbles contains 4 yellow-green and 3 blue-magenta marbles. Three marbles are drawn randomly without replacement. Draw a probability distribution for X: number of yellow-green marble.

(ii) Interval of values

Interval of values which the variable take is considered in discrete random variable.

Examples:
$P(X = 1)$: the probability that X is equivalent to 1
$P(X > 1)$: the probability that X is more than 1
$P(X \geq 1)$: the probability that X is 1 and above
$P(X < 1)$: the probability that X is less than 1
$P(X \leq 1)$: the probability that X is 1 and below
$P(1 < X < 5)$: the probability that X is between 1 and 5
$P(1 \leq X \leq 5)$: the probability that X is at least 1 and no more than 5
$P(1 < X \leq 5)$: the probability that X is more than 1 and no more than 5
$P(1 \leq X < 5)$: the probability that X is at least 1 and less than 5
$P(X \leq 5 | X \geq 1)$: the probability that there is 5 and below given that there is at least 1..

(iii) **Expected value/ mean of *X***

Theoretical method

$$E(X) = \sum x \times P(X = x)$$

Example:
Calculate the expected value of *X* from the given table below.

x	0	1	2	3
$P(X = x)$	$\frac{3}{20}$	$\frac{7}{20}$	$\frac{5}{20}$	$\frac{5}{20}$

Experimental method

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Example:
Calculate the mean score below.

x	0	1	2	3
Frequency	3	7	5	5

Application problem of expected value/ mean

Example:
At a school carnival, Linda is in charge of operating a game stall. The table below shows the prize offered per attempt and its respective probability.

Prize	\$20	\$5	\$1
Probability	0.001	0.01	0.5

Find

- The expected profit per game for Linda if each game cost \$1.

- A customer played the game and paid $3, find his expected profit/loss.

- How much Linda charge per game if she made a profit of $200 from 150 games?

(iv) **Variance**

Formulas:

$$Var(X) = \sum (x-\mu)^2 \times P(X=x)$$

Or

$$Var(X) = E(X^2) - [E(X)]^2$$

Example:

x	0	1	2	3
$P(X=x)$	0.4	0.3	0.1	0.2

Calculate the variance from the probability distribution above.

(v) Standard deviation

Formula:

$$Std(X) = \sqrt{Var(X)}$$

Example:

x	0	1	2	3
$P(X = x)$	0.4	0.3	0.1	0.2

Calculate the standard deviation from the probability distribution above.

(vi) Effect of $\times a$ and $+b$

Expected value,

$$E(aX + b) = aE(X) + b$$

Variance

$$Var(aX + b) = a^2 Var(X)$$

Standard deviation

$$\begin{aligned}\sqrt{Var(aX + b)} &= \sqrt{a^2 Var(X)} \\ &= |a|\sqrt{Var(X)}\end{aligned}$$

Example 1:

A spinning wheel has four equal sections. Each section states the prize won by a contestant namely $1, $2 and $3 If the pay-out is changed to $0, $2 and $4, find the expected value after the change by first determining the expected value of the original prize to be given and then determine a linear rule $Y = aX + b$ to determine the new expected value.

Example 2:

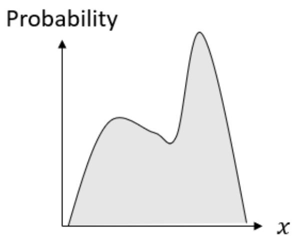
Find $E[(4X-3)]^2$ given that $E(X)=\frac{1}{2}$ and $E(X^2)=\frac{5}{10}$

(b) Continuous random variable

Definition: a variable which takes any values over intervals and whose values (measurable values) are obtained by measuring.

Example:

- Weight of elephants in the national zoo
- Height of green been seedlings after a week
- Diameter of skull



*any graph that has a region enclosed under it

(i) Probability density function

Definition: Probability density function is a function whose value at any given sample in the sample space can be interpreted as providing a relative likelihood that the value of the random variable would equal that sample.

Probability density function (characteristics):

- Sum (total) of the probabilities is 1: $\int_b^a f(x)\,dx = 1$
- $f(x) \geq 0$ for interval $a \leq x \leq b$
- $P(X=k)=0$ *which is same as* $\int_k^k f(x)\,dx = 0$
- $P(X \leq k) = P(X<k) + P(X=k)$
  $= P(X<k)$

Finding probability density function given a cumulative distribution function

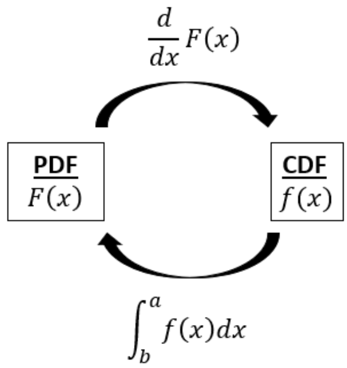
$$f(x)=\frac{d}{dx}F(x)$$

(ii) **Cumulative distribution function**

Definition: Cumulative distribution function expresses the probability that X does not exceed the value of x.

$$F(x) = P(X \leq x)$$
$$= \int_{-\infty}^{x} f(x)\,dx$$

(iii) **PDF and CDF**



Example:
The probability density function f of a continuous random variable T is given by,

$$f(t) = \begin{cases} \frac{1}{24}t & 0 \leq t \leq 4 \\ \frac{1}{4} - \frac{1}{48}t & 4 \leq t \leq 12 \\ 0 & \text{Otherwise} \end{cases}$$

- Find the cumulative distribution function for T.

- Find $P(3 < T < 12)$.

(iv) Expected value/ mean of *X*

$$E(X) = \int_a^b x \times f(x)\,dx$$

Example:

Random variable X has a probability function of $f(x) = \frac{e^x}{2}$ for $0 \le x \le ln\,3$. Determine the mean for X.

(v) Variance

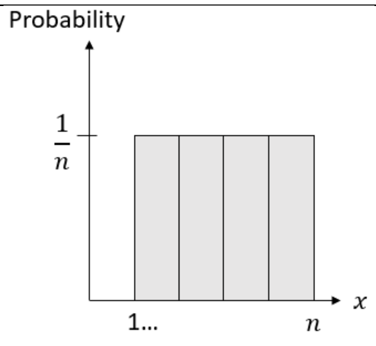
$$Var(X) = \int_a^b x^2 \times f(x)\,dx - [E(X)]^2$$

Example:

Random variable X has a probability function of $f(x) = 2x$ for $0 \le x \le 1$. Determine the variance for X.

3. Uniform distribution

(a) Discrete uniform distribution



Discrete uniform variable properties (characteristics):

- n values in the range has equal probability $\frac{1}{n}$
  (the probability of uniformly spaced possible values is equal)

Probability mass function:

$$P(X = x) = \frac{1}{n} \text{ for } x = 1,2,3,4 \ldots., n$$

Notation:

$$X \sim U\{a, b\}$$

	(i)	**Mean/ expected value** Formula: $$E(X) = \frac{n+1}{2}$$ Derivation of formula: $E(X) = \sum x \ P(X = x)$ $= 1\left(\frac{1}{n}\right) + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + 4\left(\frac{1}{n}\right) + \cdots . + n\left(\frac{1}{n}\right) *$ $= \frac{1+n}{2}$ $* \ S_n = \frac{n}{2}[2a + (n-1)d]$ $= \frac{n}{2}\left[2\left(\frac{1}{n}\right) + (n-1)\left(\frac{1}{n}\right)\right]$ $= \frac{n}{2}\left[\frac{2+n-1}{n}\right]$ $= \frac{n+n^2}{2n}$ $= \frac{1+n}{2}$
		Example 1: Given that a fair die is rolled and let Z implies that a number appears on the face of the die. Find the mean of Z. Example 2: Given that a discrete uniform variable is given by $U\{1,7\}$ and that $P(X = x) = \frac{1}{7}$. Find the value of the mean.
	(ii)	**Variance** Formula: $$Var(X) = \frac{n^2-1}{12}$$

Derivation of formula:

$Var(X) = E(X^2) - [E(X)]^2$ *

$$= \frac{(n+1)(2n+1)}{6} - (\frac{1+n}{2})^2$$

$$= \frac{2n^2+n+2n+1}{6} - (\frac{n^2+2n+1}{4})$$

$$= \frac{4n^2+6n+2-(3n^2+6n+3)}{12}$$

$$= \frac{n^2-1}{12}$$

$* E(X^2) = \sum x^2 \times P(X = x)$

$$= 1^2\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + 3^2\left(\frac{1}{n}\right) + 4^2\left(\frac{1}{n}\right) + \cdots . + n^2\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}[1^2 + 2^2 + 3^2 + 4^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$[E(X)]^2 = (\frac{1+n}{2})^2$$

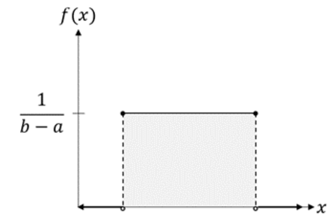
Example 1:

Suppose that a discrete uniform variable is given by $U\{1,7\}$. Find the value of the variance.

Example 2:

k and h each has a discrete distribution which is uniform for the integers $1,2,3,4, \ldots . n$. Show that $Var(k) + Var(h) = \frac{n^2-1}{6}$.

(b) Continuous uniform distribution



Probability density function:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Notation:

$$X \sim U(a, b) \text{ if } a < x < b$$
$$X \sim U[a, b] \text{ if } a \le x \le b$$

(i) Mean/ expected value

Formula:

$$E(X) = \frac{a+b}{2}$$

Derivation of formula:

$$E(X) = \int_a^b x\left(\frac{1}{b-a}\right) dx$$
$$= [\frac{x^2}{2(b-a)}]_a^b$$
$$= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)}$$
$$= \frac{b^2 - a^2}{2(b-a)}$$
$$= \frac{(b+a)(b-a)}{2(b-a)}$$
$$= \frac{a+b}{2}$$

Example 1:
A continuous random variable X is uniformly distributed in the interval $2 \le x \le 10$.
Find $E(X)$.

Example 2:
The radius of a circle drawn, R can be any value between 5.5cm and 7.5cm. State the mean for R.

(ii) **Variance**

Formula:

$$Var(X) = \frac{(b-a)^2}{12}$$

Derivation of formula:

$$Var(X) = \int_a^b x^2\left(\frac{1}{b-a}\right)dx - \left(\frac{a+b}{2}\right)^2$$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b - \frac{(a+b)^2}{4}$$

$$= \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)(b^2+ba+a^2)}{3(b-a)} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{(b^2+ba+a^2)}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4(b^2+ba+a^2)}{12} - \frac{3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Example 1:
A continuous random variable X is uniformly distributed in the interval $84 \le x \le 90$. Find $Var(X)$.

		Example 2: The speed of cars on the road, V can be any values in the interval, $30 \leq V \leq 80$. Find the variance of V.
	(iii)	**Cumulative density function** Formula: $$P(X \leq x) = \frac{x-a}{b-a}$$ Derivation of formula: $P(X \leq x) = \int_a^x \left(\frac{1}{b-a}\right) dx$ $= [\frac{x}{b-a}]_a^x$ $= \frac{x}{b-a} - \frac{a}{b-a}$ $= \frac{x-a}{b-a}$

END